Derivation of Hamiltonian and Hamilton's Equations for a Magnetic Moment in a Magnetic Field

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Initial Lagrangian

Assume we have a given spin $\vec{S}$ in a magnetic field $\vec{B}$ with magnetic moment μ. Thus we can find the energy of the system by the following Lagrangian:

$$L = \mu \vec{B} \cdot \vec{S}$$

We can then augment the kinetic energy term to arrive at:

$$H_{aug} = \frac{1}{2} m \|\dot{\vec{S}}\|^2 + \mu \vec{B} \cdot \vec{S}$$

where $\vec{S} = (S_x, S_y, S_z)$ and $\vec{B} = (0, 0, B)^T$, and the constraint $\|S\| = 1$.

Transforming to Spherical Coordinates

We will now express the system in spherical coordinates, which are ideal for representing a spin vector's angular properties and directionality in three-dimensional space. Expressing $\vec{S}$ in spherical coordinates:

$$\vec{S} = S \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

The kinetic energy term in spherical coordinates is derived from $\dot{\vec{S}}$:

$$\dot{\vec{S}} = S \begin{pmatrix} \cos(\theta)\cos(\phi)\dot{\theta} - \sin(\theta)\sin(\phi)\dot{\phi} \\ \cos(\theta)\sin(\phi)\dot{\theta} + \sin(\theta)\cos(\phi)\dot{\phi} \\ -\sin(\theta)\dot{\theta} \end{pmatrix}$$

Thus, $\|\dot{\vec{S}}\|^2$ becomes:

$$\|\dot{\vec{S}}\|^2 = \dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2$$

And we can simplify potential energy to:

$$\mu\vec{B}\cdot\vec{S} = \mu B\cos(\theta)$$

Now that we have our velocity vector generalized in spherical coordinates, with some computation we can simplify our Lagrangian to:

$$L = \frac{1}{2}m\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right) + \mu B\cos(\theta)$$

Deriving Conjugate Momenta and Generalized Velocities

Now, I will change our notation and derive a Hamiltonian from our Lagrangian so we can solve for our equations of motion in terms of conjugate momenta and generalize velocities. We know that the conjugate momenta and generalized velocities are found from our Lagrangian as:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = m\sin^2(\theta)\dot{\phi}$$

$$\dot{\theta} = \frac{p_\theta}{m}$$

$$\dot{\phi} = \frac{p_\phi}{m\sin^2(\theta)}$$

We can then substitute these into the Hamiltonian to get the following:

$$H = \frac{p_\theta^2}{2m} + \frac{p_\phi^2}{2m\sin^2(\theta)} + \mu B\cos(\theta)$$

Hamilton's Equations

Now in order to get our first order differential equations for this system we can use Hamilton's equations and solve for the following generalized velocities:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial\theta} = -\frac{p_\phi^2\cos(\theta)}{m\sin^3(\theta)} + \mu B\sin(\theta)$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\sin^2(\theta)}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial\phi} = 0$$